2024 Kansas Collegiate Math Competition

Solutions

1. Please find the largest value of $n \in \mathbb{N}$ so that 21^n divides $2024!$

Solution:

Since $21 = 3 \cdot 7$, we need to find the exponent on 7 in the prime factorization of $2024!$. This is given by Legendre's Formula

$$\left\lfloor \frac{2024}{7} \right\rfloor + \left\lfloor \frac{2024}{7^2} \right\rfloor + \left\lfloor \frac{2024}{7^3} \right\rfloor = 289 + 41 + 5 = 335.$$

This works because there are $\lfloor \frac{2024}{7} \rfloor$ numbers in the range $1, 2, \cdots 2024$ divisible by 7. Each of these numbers contributes at least a single 7 to the prime factorization of $2024! = 1 \cdot 2 \cdots 2024$. There are $\lfloor \frac{2024}{7^2} \rfloor$ numbers in the range divisible by 49 (and each of these contribute an additional 7), and so on and so forth. The same method can be used to compute the exponent on 3 (which will result in a larger exponent). The smaller of the two exponents on $3, 7$ determine divisibility by 21. Therefore, the largest value of n is 335.

2. Suppose that

$$3 = \frac{2}{x_1} = x_1 + \frac{2}{x_2} = x_2 + \frac{2}{x_3} = x_3 + \frac{2}{x_4} = \dots$$

Guess an expression, in terms of n, for x_n. Then, **prove rigorously** the correctness of your guess.

Solution:

Note that each equality can be written as

$$3 = x_n + \frac{2}{x_{n+1}} \;\Leftrightarrow\; x_{n+1} = \frac{2}{3 - x_n}.$$

We find $x_1 = \frac{2}{3}, x_2 = \frac{6}{7}, x_3 = \frac{14}{15}, x_4 = \frac{30}{31}$. Thus, our guess is

$$x_n = \frac{2^{n+1} - 2}{2^{n+1} - 1}.$$

In order to establish this expression rigorously, we use induction. The expression is true for $n = 1$. Assuming it holds for n, we look at x_{n+1}. From the recursion relation above, we have

$$x_{n+1} = \frac{2}{3 - x_n} = \frac{2}{3 - \frac{2^{n+1}-2}{2^{n+1}-1}} = \dots = \frac{2^{n+2} - 2}{2^{n+2} - 1}$$

as desired, thereby concluding the proof.

3. A random number generator selects one of the nine integers $1, 2, \ldots 9$ with uniform probability. Please determine the probability that after n selections, the product of the n numbers will be divisible by 10.

Solution:

The product of the n numbers will be divisible by 10 if and only if at least one even number is selected and 5 is selected at least once. Let

$$\begin{aligned} A = &\text{ event at least one even number is selected} \\ B = &\text{ event 5 is selected at least once} \end{aligned}$$

We are trying to find $P(A \cap B)$. It is easier to calculate the probability of the complement

$$P((A \cap B)^C) = P(A^C \cup B^C) = P(A^C) + P(B^C) - P(A^C \cap B^C).$$

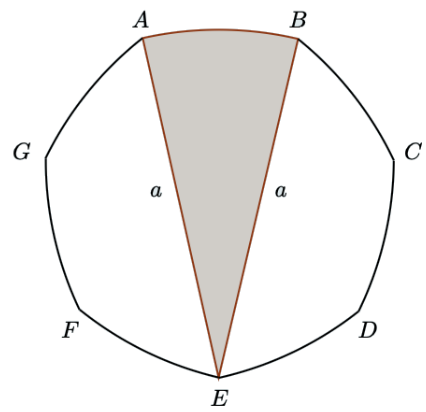
Now observe that

$$\begin{aligned} P(A^C) &= P(\text{all the numbers are one of the 5 odd numbers}) = \frac{5^n}{9^n} \\ P(B^C) &= P(\text{none of the numbers are 5}) = \frac{8^n}{9^n} \\ P(A^C \cap B^C) &= P(\text{all of the numbers are one of 1, 3, 7, 9}) = \frac{4^n}{9^n}. \end{aligned}$$

Therefore, $P((A \cap B)^C) = \frac{5^n}{9^n} + \frac{8^n}{9^n} - \frac{4^n}{9^n}$, and hence

$$P(A \cap B) = 1 - \frac{5^n}{9^n} - \frac{8^n}{9^n} + \frac{4^n}{9^n}.$$

4. The diagram below shows what is known as a (regular) Reuleaux heptagon. The seven arcs $AB, BC, \ldots, FG, GA$ are of equal length and each arc is formed from the circle of radius a having its center at the vertex diametrically opposite the midpoint of the arc. Compute the area of the Reuleaux heptagon (in terms of a).



Solution:

In the figure, the point O is equidistant from each of three vertices A, B and E. The plan is to find the area of the sector AOB by calculating the area of AEB and subtracting the areas of the two congruent isosceles triangles OBE and OAE. The required area is 7 times this. First we need the angle AEB. We know that the angle $AOB = \frac{2}{7}\pi$ and hence the angle $BOE = \frac{1}{2}(2\pi - \frac{2}{7}\pi) = 6\pi$ (using the sum of angles round the point O). Finally, the angle $AEB = 2OEB = 2 \cdot \frac{1}{2}(\pi - \frac{6}{7}\pi) = \frac{1}{7}\pi$ using the sum of angles of an isosceles triangle.

Now ABE is a sector of a circle of radius a, so its area is

$$\pi a^2 \cdot \frac{\frac{1}{7}\pi}{2\pi} = \frac{\pi a^2}{14}.$$

The area of the triangle OBE is $\frac{1}{2}BE \cdot$ height i.e.

$$\frac{1}{2}a \cdot \frac{1}{2}a \tan(\text{angle } OBE) = \frac{a^2}{4} \tan\frac{\pi}{14}.$$

The area of the heptagon is therefore

$$7 \cdot \left(\frac{\pi a^2}{14} - 2 \cdot \frac{a^2}{4} \tan\frac{\pi}{14}\right) = \frac{a^2}{2}\left(\pi - 7\tan\frac{\pi}{14}\right).$$

5. Suppose that for $n \in \mathbb{N}$, $n = 1 \mod 4$ or $n = 2 \mod 4$. Please prove that

$$\int_0^\pi \cos(x) \cdot \cos(2x) \cdots \cos(nx)\ dx = 0.$$

Solution:

We first apply a substitution $u = x - \frac{\pi}{2}$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\left(u + \frac{\pi}{2}\right) \cdot \cos\left(2\left(u + \frac{\pi}{2}\right)\right) \cdots \cos\left(n\left(u + \frac{\pi}{2}\right)\right)\ dx.$$

Observe that

$$\cos\left(k\left(u + \frac{\pi}{2}\right)\right) = \cos(ku)\cos\left(\frac{k\pi}{2}\right) - \sin(ku)\sin\left(\frac{k\pi}{2}\right).$$

Now there are 4 cases, depending on what k is congruent to mod 4.

$$\begin{aligned}
k = 0 \mod 4 &: \cos\left(k\left(u + \frac{\pi}{2}\right)\right) = \cos(ku) \\
k = 1 \mod 4 &: \cos\left(k\left(u + \frac{\pi}{2}\right)\right) = -\sin(ku) \\
k = 2 \mod 4 &: \cos\left(k\left(u + \frac{\pi}{2}\right)\right) = -\cos(ku) \\
k = 3 \mod 4 &: \cos\left(k\left(u + \frac{\pi}{2}\right)\right) = \sin(ku)
\end{aligned}$$

Thus the terms in the product $\cos\left(u + \frac{\pi}{2}\right) \cdot \cos\left(2\left(u + \frac{\pi}{2}\right)\right) \cdots \cos\left(n\left(u + \frac{\pi}{2}\right)\right)$ alternate between these 4 cases (starting with $k = 1$). The resulting product will be an odd function precisely when $\sin$ occurs an odd number of times in the product. Clearly this only occurs when $n = 1 \mod 4$ or $n = 2 \mod 4$.

6. Frosty the snowman is made from two uniform spherical snowballs, initially of radii $2R$ and $3R$. The smaller (which is the head) stands on top of the larger. As each snowball melts, its volume decreases at a rate which is directly proportional to its surface area, the constant of proportionality being the same for each snowball. During melting, the snowballs remain spherical and uniform.

(i) When Frosty is half its initial height, show that the ratio of its volume to its initial volume is $37 : 224$.

(ii) What is this ratio when Frosty is one tenth of its initial height?

Solution: Based on the rate of change of the volume of each sphere, we have

$$\frac{d}{dt}\frac{4}{3}\pi r^3 = k \cdot 4\pi r^2$$

where r is the radius of the sphere and k is a (negative) constant which is the same for both spheres. Thus,

$$\frac{dr}{dt} = k \;\Rightarrow\; r = kt + r_0$$

where $r_0 = r(0)$ is the initial value of r.

Now, for the head we have $r_0 = 2R$ while for the body $r_0 = 3R$. Hence, we find

$$r_{\text{head}} = kt + 2R, \quad r_{\text{body}} = kt + 3R.$$

The snowman's height is

$$h = 2\left(r_{\text{head}} + r_{\text{body}}\right) = 4kt + 10R.$$

So, $h(0) = 10R$ and hence the height becomes half the initial one when

$$4kt + 10R = 5R \;\Rightarrow\; kt = -\frac{5}{4}R.$$

Then, the ratio of volumes is

$$\frac{r_{\text{head}}(-\frac{5}{4}R/k)^3 + r_{\text{body}}(-\frac{5}{4}R/k)^3}{r_{\text{head}}(0)^3 + r_{\text{body}}(0)^3} = \frac{(\frac{3}{4}R)^3 + (\frac{7}{4}R)^3}{(2R)^3 + (3R)^3} = \frac{27 + 343}{64 \cdot 35} = \frac{37}{224}.$$

Moreover, when $h = h(0)/10 = R$ it is important to note that the formula for the height implies

$$4kt + 10R = R \;\Rightarrow\; kt = -\frac{9}{4}R.$$

At this value of kt, however, the radius of the head would be negative. Indeed, the head melts fully at $kt = -2R$ and after that point the height of the snowman is just twice the radius of the body. Hence, when the snowman has height R it must be that $r_{\text{body}} = \frac{1}{2}R$ and then the ratio of the volumes is

$$\frac{r_{\text{body}}^3}{r_{\text{head}}(0)^3 + r_{\text{body}}(0)^3} = \frac{(\frac{R}{2})^3}{(2R)^3 + (3R)^3} = \frac{1}{8 \cdot 35} = \frac{1}{280}.$$

7. Please prove that coefficient of a^k in the expansion of $(1+a+a^2+a^3)^n$ is

$$\sum_{i=0}^{k}\binom{n}{i}\binom{n}{k-2i}$$

(assume that $\binom{n}{r}=0$ for $n \in \mathbb{N}$ and $r<0$)

Solution:

Rewriting and applying Binomial Theorem

$$\begin{aligned}(1+a+a^2+a^3)^n &= ((1+a)+a^2(1+a))^n \\ &= (1+a)^n(1+a^2)^n.\end{aligned}$$

All powers of a in the expansion of $(1+a^2)^n$ are even. A typical term in the expansion of $(1+a^2)^n$ is of the form

$$\binom{n}{i}a^{2i}.$$

This will multiply with a term of the form

$$\binom{n}{k-2i}a^{k-2i}$$

in the expansion of $(1+a)^n$ to make an a^k term (but only for $k-2i \geq 0$). Adding up the coefficients of all such parings yields the desired equality.

8. Let $f(x)$ be a continuous function defined for $x \in [0,1]$ and a be a real number such that

$$\int_0^1 e^{ax}\left(f(x)\right)^2 dx = 2\int_0^1 f(x)dx + \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4}.$$

Find a and $f(x)$.

Solution: Starting from the given relation, we complete the square to get

$$\int_0^1 \left(e^{\frac{ax}{2}}f(x) - e^{-\frac{ax}{2}}\right)^2 dx = -\int_0^1 e^{-ax}dx + \frac{e^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4}$$

or, computing the integral on the right-hand side,

$$\int_0^1 \left(e^{\frac{ax}{2}}f(x) - e^{-\frac{ax}{2}}\right)^2 dx = \frac{1}{a} - \frac{1}{a^2} - \frac{1}{4} = \frac{4a-4-a^2}{4a^2} = -\frac{(a-2)^2}{4a^2} \leq 0.$$

Thus, since $\int_0^1 \left(e^{\frac{ax}{2}}f(x) - e^{-\frac{ax}{2}}\right)^2 dx \geq 0$, we infer

$$\int_0^1 \left(e^{\frac{ax}{2}}f(x) - e^{-\frac{ax}{2}}\right)^2 dx = -\frac{(a-2)^2}{4a^2} = 0.$$

Hence, $a=2$. Moreover, since $\left(e^{\frac{ax}{2}}f(x) - e^{-\frac{ax}{2}}\right)^2 \geq 0$, it must be (using also continuity) that

$$\left(e^{\frac{ax}{2}}f(x) - e^{-\frac{ax}{2}}\right)^2 = 0 \quad \forall x \in [0,1]$$

i.e. $f(x) = e^{-ax} = e^{-2x}$.

9. For each positive integer n, let

$$a_n = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots$$

(i) Prove that $0 < a_n < \frac{1}{n}$.

(ii) Show that $a_n = n!e - \lfloor n!e \rfloor$, where $\lfloor x \rfloor \in \mathbb{Z}$ denotes the integer part (or floor) of $x > 0$.

(iii) Hence, prove that e is irrational.

Solution: For (i), it is immediate that $a_n > 0$ and also we observe that

$$a_n < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots = \sum_{j=1}^{\infty} \left(\frac{1}{n+1}\right)^j = \frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} = \frac{1}{n}.$$

For (ii), we note that, from (i), $a_n \in (0,1)$. In addition, from the Taylor series for e^x we have

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} + \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots$$

and so

$$n!e = n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \dots + 1 + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \dots$$

i.e. $n!e = n! + n! + \frac{n!}{2!} + \frac{n!}{3!} + \dots + 1 + a_n$. Since all of the terms on the right-hand side except for a_n are integers, and since $a_n \in (0,1)$, it follows that $a_n = n!e - \lfloor n!e \rfloor$.

Finally, for (iii), suppose to the contrary that $e = k/m$ where $k, m \in \mathbb{Z}$. Then, $m!e$ is an integer and so $\lfloor m!e \rfloor = m!e$. But then $m!e - \lfloor m!e \rfloor = 0$, which contradicts (ii) since $a_m \in (0,1)$.

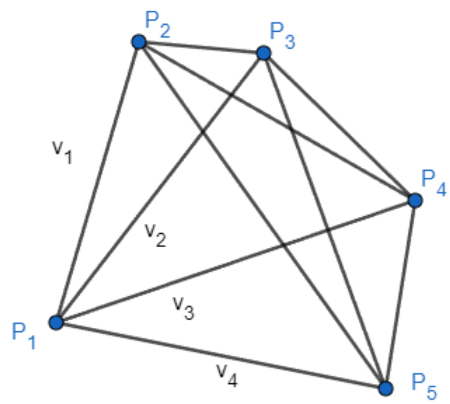
10. Let P_1, P_2, P_3, P_4, P_5 by *any* 5 points in the plane $\mathbb{R}^2$ and A be the 5×5 matrix whose $(i,j)-$entry a_{ij} is the **square** of the distance from P_i to P_j (given by the standard distance formula). Please prove that the determinant of A is 0

$$\det(A) = 0.$$

hint: interpret the problem in terms of vectors.

Solution:

Let $\mathbf{v}_i$ denote the vector from P_1 to P_i, $1 \le i \le 4$.



Then the distance from P_2 to P_3 is the magnitude of $\mathbf{v}_1 - \mathbf{v}_2$, and so on. Recalling that $|\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u}$, we see that A can be expressed as

$$A = \begin{bmatrix} 0 & \mathbf{v}_1 \cdot \mathbf{v}_1 & \mathbf{v}_2 \cdot \mathbf{v}_2 & \mathbf{v}_3 \cdot \mathbf{v}_3 & \mathbf{v}_4 \cdot \mathbf{v}_4 \\ \mathbf{v}_1 \cdot \mathbf{v}_1 & 0 & (\mathbf{v}_1 - \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) & (\mathbf{v}_1 - \mathbf{v}_3) \cdot (\mathbf{v}_1 - \mathbf{v}_3) & (\mathbf{v}_1 - \mathbf{v}_4) \cdot (\mathbf{v}_1 - \mathbf{v}_4) \\ \mathbf{v}_2 \cdot \mathbf{v}_2 & (\mathbf{v}_1 - \mathbf{v}_2) \cdot (\mathbf{v}_1 - \mathbf{v}_2) & 0 & (\mathbf{v}_2 - \mathbf{v}_3) \cdot (\mathbf{v}_2 - \mathbf{v}_3) & (\mathbf{v}_2 - \mathbf{v}_4) \cdot (\mathbf{v}_2 - \mathbf{v}_4) \\ \mathbf{v}_3 \cdot \mathbf{v}_3 & (\mathbf{v}_1 - \mathbf{v}_3) \cdot (\mathbf{v}_1 - \mathbf{v}_3) & (\mathbf{v}_2 - \mathbf{v}_3) \cdot (\mathbf{v}_2 - \mathbf{v}_3) & 0 & (\mathbf{v}_3 - \mathbf{v}_4) \cdot (\mathbf{v}_3 - \mathbf{v}_4) \\ \mathbf{v}_4 \cdot \mathbf{v}_4 & (\mathbf{v}_1 - \mathbf{v}_4) \cdot (\mathbf{v}_1 - \mathbf{v}_4) & (\mathbf{v}_2 - \mathbf{v}_4) \cdot (\mathbf{v}_2 - \mathbf{v}_4) & (\mathbf{v}_3 - \mathbf{v}_4) \cdot (\mathbf{v}_3 - \mathbf{v}_4) & 0 \end{bmatrix}$$

Expanding the dot products (and writing $\mathbf{v}_i \cdot \mathbf{v}_i$ as $\mathbf{v}_i^2$),

$$A = \begin{bmatrix} 0 & \mathbf{v}_1^2 & \mathbf{v}_2^2 & \mathbf{v}_3^2 & \mathbf{v}_4^2 \\ \mathbf{v}_1^2 & 0 & \mathbf{v}_1^2 - 2\mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_2^2 & \mathbf{v}_1^2 - 2\mathbf{v}_1\mathbf{v}_3 + \mathbf{v}_3^2 & \mathbf{v}_1^2 - 2\mathbf{v}_1\mathbf{v}_4 + \mathbf{v}_4^2 \\ \mathbf{v}_2^2 & \mathbf{v}_1^2 - 2\mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_2^2 & 0 & \mathbf{v}_2^2 - 2\mathbf{v}_2\mathbf{v}_3 + \mathbf{v}_3^2 & \mathbf{v}_2^2 - 2\mathbf{v}_2\mathbf{v}_4 + \mathbf{v}_4^2 \\ \mathbf{v}_3^2 & \mathbf{v}_1^2 - 2\mathbf{v}_1\mathbf{v}_3 + \mathbf{v}_3^2 & \mathbf{v}_2^2 - 2\mathbf{v}_2\mathbf{v}_3 + \mathbf{v}_3^2 & 0 & \mathbf{v}_3^2 - 2\mathbf{v}_3\mathbf{v}_4 + \mathbf{v}_4^2 \\ \mathbf{v}_4^2 & \mathbf{v}_1^2 - 2\mathbf{v}_1\mathbf{v}_4 + \mathbf{v}_4^2 & \mathbf{v}_2^2 - 2\mathbf{v}_2\mathbf{v}_4 + \mathbf{v}_4^2 & \mathbf{v}_3^2 - 2\mathbf{v}_3\mathbf{v}_4 + \mathbf{v}_4^2 & 0 \end{bmatrix}$$

Apply the sequence of row operations $R_i = -R_1 + R_i$ for $1 \le i \le 4$ (leaving the determinant unchanged).

$$\begin{bmatrix} 0 & \mathbf{v}_1^2 & \mathbf{v}_2^2 & \mathbf{v}_3^2 & \mathbf{v}_4^2 \\ \mathbf{v}_1^2 & -\mathbf{v}_1^2 & \mathbf{v}_1^2 - 2\mathbf{v}_1\mathbf{v}_2 & \mathbf{v}_1^2 - 2\mathbf{v}_1\mathbf{v}_3 & \mathbf{v}_1^2 - 2\mathbf{v}_1\mathbf{v}_4 \\ \mathbf{v}_2^2 & -2\mathbf{v}_1\mathbf{v}_2 + \mathbf{v}_2^2 & -\mathbf{v}_2^2 & \mathbf{v}_2^2 - 2\mathbf{v}_2\mathbf{v}_3 & \mathbf{v}_2^2 - 2\mathbf{v}_2\mathbf{v}_4 \\ \mathbf{v}_3^2 & -2\mathbf{v}_1\mathbf{v}_3 + \mathbf{v}_3^2 & -2\mathbf{v}_2\mathbf{v}_3 + \mathbf{v}_3^2 & -\mathbf{v}_3^2 & \mathbf{v}_3^2 - 2\mathbf{v}_3\mathbf{v}_4 \\ \mathbf{v}_4^2 & -2\mathbf{v}_1\mathbf{v}_4 + \mathbf{v}_4^2 & -2\mathbf{v}_2\mathbf{v}_4 + \mathbf{v}_4^2 & -2\mathbf{v}_3\mathbf{v}_4 + \mathbf{v}_4^2 & -\mathbf{v}_4^2 \end{bmatrix}$$

Now apply the sequence of column operations $C_i = -C_1 + C_i$ for $1 \leq i \leq 4$ (again leaving the determinant unchanged).

$$\begin{bmatrix} 0 & \mathbf{v}_1^2 & \mathbf{v}_2^2 & \mathbf{v}_3^2 & \mathbf{v}_4^2 \\ \mathbf{v}_1^2 & -2\mathbf{v}_1^2 & -2\mathbf{v}_1\mathbf{v}_2 & -2\mathbf{v}_1\mathbf{v}_3 & -2\mathbf{v}_1\mathbf{v}_4 \\ \mathbf{v}_2^2 & -2\mathbf{v}_1\mathbf{v}_2 & -2\mathbf{v}_2^2 & -2\mathbf{v}_2\mathbf{v}_3 & -2\mathbf{v}_2\mathbf{v}_4 \\ \mathbf{v}_3^2 & -2\mathbf{v}_1\mathbf{v}_3 & -2\mathbf{v}_2\mathbf{v}_3 & -2\mathbf{v}_3^2 & -2\mathbf{v}_3\mathbf{v}_4 \\ \mathbf{v}_4^2 & -2\mathbf{v}_1\mathbf{v}_4 & -2\mathbf{v}_2\mathbf{v}_4 & -2\mathbf{v}_3\mathbf{v}_4 & -2\mathbf{v}_4^2 \end{bmatrix}$$

The maximum number of linear independent vectors is 2 in $\mathbb{R}^2$, so there is no loss of generality in assuming that $\mathbf{v}_3$ and $\mathbf{v}_4$ are linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ (swapping rows if necessary). So there exists $a, b, c, d \in \mathbb{R}$ so that $\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2$ and $\mathbf{v}_4 = c\mathbf{v}_1 + d\mathbf{v}_2$. Then after we apply the row operations $R_3 = -aR_1 - bR_2 + R_3$ and $R_4 = -cR_1 - dR_2 + R_4$, our matrix has the form

$$\begin{bmatrix} 0 & \mathbf{v}_1^2 & \mathbf{v}_2^2 & \mathbf{v}_3^2 & \mathbf{v}_4^2 \\ \mathbf{v}_1^2 & -2\mathbf{v}_1^2 & -2\mathbf{v}_1\mathbf{v}_2 & -2\mathbf{v}_1\mathbf{v}_3 & -2\mathbf{v}_1\mathbf{v}_4 \\ \mathbf{v}_2^2 & -2\mathbf{v}_1\mathbf{v}_2 & -2\mathbf{v}_2^2 & -2\mathbf{v}_2\mathbf{v}_3 & -2\mathbf{v}_2\mathbf{v}_4 \\ * & 0 & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \end{bmatrix}.$$

To see this, note that a generic entry in the 3rd row becomes

$$\begin{aligned} 2a\mathbf{v}_1 \cdot \mathbf{v}_i + 2b\mathbf{v}_2 \cdot \mathbf{v}_i - 2\mathbf{v}_3 \cdot \mathbf{v}_i &= 2((a\mathbf{v}_1 + b\mathbf{v}_2 - \mathbf{v}_3) \cdot \mathbf{v}_i) \\ &= \mathbf{0} \cdot \mathbf{v}_i = 0, \end{aligned}$$

and similarly for the entries in the 4th row. Now we can infer a vanishing determinant by cofactor expansion along the 4th or 5th row.

.